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CHAPTER

High Energy Particles 2

Theory of Electromagnetic Interactions

2.1. ***General remarks.*** Theoretical physicists have not yet succeeded in their attempts to formulate the principles of quantum electrodynamics in a completely general manner, free from internal contradictions. They have, however, established a formalism that answers unambiguously most problems arising in the study of electromagnetic interactions between radiation and matter. Whenever the theoretical predictions have been submitted to experimental test, they have been found to be accurate, within the limits of the experimental errors and the mathematical approximations made in the development of the theory. Confidence in the theory of electromagnetic interactions has grown to the point where one may grant its validity beyond the limits of experimental accuracy and perhaps even apply it to fields where experimental tests are still lacking. In the past, study of high-energy phenomena, cosmic rays in particular, was mainly a means for testing the theory of electromagnetic interactions. Today, however, one may justifiably use the results of this theory as a basis for the interpretation of the observed phenomena.

A rigorous derivation of the theoretical formulae lies beyond the scope of the present volume. In many cases, however, we shall try to justify these formulae by means of semi-quantitative derivations based largely on classical models. This procedure provides a physical interpretation for the laws expressed by the theoretical formulae and thus develops an intuitive "feeling" for the phenomena associated with the passage of high-energy particles through matter. We believe that this purpose is important, because one must often rely on such intuition to grasp the significance of a set of experimental data or to devise new methods for the solution of a given problem.

In the study of electromagnetic interactions we encounter two different kinds of entities: *electromagnetic fields* and *particles*. The classical Maxwell theory, leading to the concept of *electromagnetic waves*, fully describes the macroscopic *electromagnetic field*. In the microscopic realm, however, the field obeys quantum laws whose significance, in certain cases, we may regard as intuitive by thinking of the electromagnetic field as a flux of *photons*. The *particles*, e.g., *electrons*, *mesons*, *protons*, are both the sources of the electromagnetic field and the recipients of its effects. Most of these particles appear in a dual capacity, namely, as radiation quanta and as constituents of matter. Their electromagnetic properties depend on their electric charges and magnetic moments. Their mechanical properties depend on their masses and their spins.

In the rigorous sense, we should always treat the interactions between two particles in terms of the electromagnetic fields set up by the particles and the effects of these fields on the particles themselves. This remark applies to both classical and quantum electrodynamics. If we look at the corpuscular aspect of the electromagnetic field, we may say that electromagnetic interactions should always be described as processes of photon emission and absorption. However, many cases arise in classical electrodynamics where one can calculate the interaction of two charged bodies in terms of the relatively simple Coulomb forces acting between their charges, rather than in terms of the more general electromagnetic field. Correspondingly there occur problems of quantum electrodynamics wherein one can neglect emission or absorption of photons and describe the electromagnetic interactions between particles by means of suitable fields of force. In fact, even when photons are specifically included in the formulation of the problem, one generally proceeds by first computing the mechanical behavior of the particles concerned without reference to emission or absorption of photons, and later introducing radiation phenomena as a perturbation.

With the above considerations in mind, we now proceed to a classification of the elementary electromagnetic phenomena that are of importance in the interactions with matter of high-energy radiation quanta.

Consider first the various phenomena that take place when a charged particle passes in the neighborhood of an atom.

If the distance of closest approach is large compared with the dimensions of the atom, the atom reacts as a whole to the variable field set up by the passing particle. The result is an *excitation* or an *ionization* of the atom. We can treat the phenomenon by the ordinary methods of quantum mechanics without direct reference to radiation. For these comparatively distant collisions, the magnetic moment of the particle is of secondary importance, because the forces associated with the magnetic moment decrease as the third power of the distance, whereas the Coulomb forces decrease as the square of the distance. Therefore we can consider the passing particle as a point charge.

If the distance of closest approach is of the order of the atomic dimensions, the interaction no longer involves the passing particle and the atom as a whole, but rather the passing particle and one of the atomic electrons.

As a consequence of the interaction, the electron is ejected from the atom with considerable energy. This phenomenon is often described as a knock-on process. If the energy acquired by the secondary electron is large compared with the binding energy, the phenomenon can be treated as an interaction between the passing particle and a free electron. Radiation phenomena can still be neglected, and the ordinary methods of quantum mechanics can be used. However, one can no longer neglect the magnetic moments or spins of the interacting particles. When the particles are identical (e.g., electron-electron collisions), exchange phenomena occur and acquire special importance when the minimum distance of approach becomes comparable with the deBroglie wavelength. The phenomena described above will be referred to as "non-radiative collision processes" or, more simply, "collision processes."

When the distance of closest approach becomes smaller than the atomic radius, the deflection of the trajectory of the passing particle in the electric field of the nucleus becomes the most important effect. Classically each deflection results in the emission of a weak electromagnetic radiation with a continuous frequency spectrum. Quantum-theoretically, a number of "soft" quanta, whose total energy is usually a very small fraction of the particle energy, accompany the deflection. In a few cases, however, one photon of energy comparable with that of the particle is emitted. Because of the comparatively small probability of this effect, we can treat the problem of the *scattering* of particles separately from that of *radiation* (or *bremsstrahlung*).

We treat the problem of scattering as a purely mechanical one, according to the methods of quantum mechanics. In this problem, we replace the actual atom by a fictitious, spherically symmetrical field of force, which coincides with the Coulomb field of the nucleus at small distances from the center of the atom, and falls off more rapidly than a Coulomb field at larger distances because of the partial shielding of the electric field of the nucleus by the planetary electrons.

The problem of computing the probability of photon emission by the passage of a charged particle through an atom (radiation probability) requires the application of quantum electrodynamics. As in the scattering problem, we still represent the atom schematically by a central field of force. However, the Hamiltonian of the system, which in the case of the scattering problem consisted of the Hamiltonian of the particle exclusively, now contains also the Hamiltonian of the electromagnetic field and a small interaction term that depends on the coordinates of both the particle and the field. This interaction term produces transitions corresponding to energy transfers between the particle and the electromagnetic field. As mentioned above, the probabilities of these transitions may be computed by the perturbation method.

If we now turn our attention to the interactions of photons with matter, we may again distinguish three cases, namely: interaction of a photon with an atom as a whole, interaction of a photon with a free electron, and interaction of a photon with the Coulomb field of the nucleus.

The interaction of a photon with an atom as a whole leads to the *photo-electric effect.* The importance of this effect in the field of high energies is negligible, so that we need not consider it in detail. The interaction of a photon with a free electron leads to the *Compton effect.* In this phenomenon the photon transfers part of its energy and momentum to the electron initially at rest. The interaction of a photon with the Coulomb field of the nucleus leads to the phenomenon of *pair production,* whereby the photon disappears and a positive and a negative electron simultaneously come into existence. For this phenomenon to occur, the energy of the photon must exceed the rest energy of the two electrons. The excess energy appears almost completely as kinetic energy of the two electrons, while the recoil of the nucleus takes care of the momentum balance.

Both Compton effect and pair production are typical quantum phenomena without classical counterpart. Their description requires the use of quantum electrodynamics along with quantum mechanics. In addition to the pair production of electrons one may envisage the possibility of pair production of heavier particles, for instance, μ-mesons. The existence of this phenomenon has not yet been established experimentally, although it appears likely on theoretical grounds.

2.2. ***Application of the conservation laws to the collision of a particle with a free electron.*** As indicated above, a close collision between a charged particle and an atomic electron is not essentially different from a collision between a charged particle and a free electron. The application of the principles of conservation of energy and momentum leads to some useful relations.

Consider the vector diagram of Fig. 1. Let m be the mass of the incident particle, p its momentum before the collision, and p'' its momentum after the collision. Let m_e be the mass of the electron, assumed to be initially at rest, p' the momentum of the electron after the collision. The corresponding kinetic energy is $E' = \sqrt{c^2p'^2 + m_e^2c^4} - m_ec^2$, where c represents the velocity of light (Appendix 2b). Let θ be the angle be-

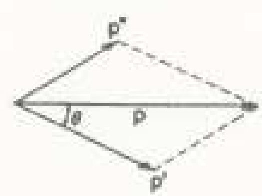

Fig. 2.2.1. Collision between a charged particle and a free electron.

tween the initial trajectory of the primary particle and the direction of motion of the electron after the collision. The principle of the *conservation of energy* gives:

$$\sqrt{p^2c^2 + m^2c^4} + m_ec^2 = \sqrt{p''^2c^2 + m^2c^4} + E' + m_ec^2 \qquad (1)$$

The conservation of momentum gives:

$$p''^2 = p'^2 + p^2 - 2pp' \cos \theta. \qquad (2)$$

Elimination of p'' between Eqs. (1) and (2) yields:

$$E' = 2m_ec^2 \frac{p^2c^2 \cos^2 \theta}{[m_ec^2 + (p^2c^2 + m^2c^4)^{1/2}]^2 - p^2c^2 \cos^2 \theta}. \qquad (3)$$

The kinetic energy, E', of the recoil electron increases with decreasing θ. The maximum transferable energy corresponds to a "head-on" collision and has the value:

$$E'_m = 2m_ec^2 \frac{p^2c^2}{m_e^2c^4 + m^2c^4 + 2m_ec^2(p^2c^2 + m^2c^4)^{1/2}}. \qquad (4)$$

For mesons and protons $m \gg m_e$, and one can neglect the term $m_e^2c^4$ in the denominator. For very large momenta ($p \gg m^2c/m_e$) Eq. (4) then becomes:

$$E'_m \approx pc \approx E. \qquad (5)$$

This result, unlike that of nonrelativistic mechanics, indicates that a particle of very high energy can transfer almost all of its kinetic energy to an electron even if the mass of the particle is much larger than the electron mass. Thus a very-high-energy meson or proton can be practically "stopped" by a head-on collision with an electron.

On the other hand, if $m \gg m_e$ and if the condition:

$$p \ll \frac{m^2c}{m_e} \qquad (6)$$

is satisfied, Eq. (4) becomes:

$$E'_m \approx 2m_ec^2 \left(\frac{p}{mc}\right)^2 = 2m_ec^2 \frac{\beta^2}{1 - \beta^2} \qquad (7)$$

where β is the velocity of the incident particle in terms of the light velocity (Appendix 2b). One sees that for heavy particles of sufficiently small momenta the maximum transferable energy depends only on the velocity.

2.3. *Theoretical expressions for the collision probabilities of charged particles with free electrons (knock-on probabilities).* Let $\Phi_{col}(E,E')\,dE'\,dx$ represent the probability for a charged particle of kinetic energy E, traversing a thickness of dx g cm^{-2}, to transfer an energy between E' and $E' + dE'$ to an atomic electron. The function Φ_{col} will be called the *differential collision probability*. In this section we shall list the theoretical expressions of Φ_{col} for electrons and for heavier particles with charge equal, in absolute value, to the electron charge, e. We shall assume that E' is sufficiently large so that the atomic electrons may be regarded as free.*

It is convenient to measure the thickness, x, in g cm^{-2} and to introduce the constant

$$C = \pi N \frac{Z}{A} r_e^2 = 0.150 \frac{Z}{A} \quad \text{g}^{-1}\,\text{cm}^2, \tag{1}$$

where Z and A are the charge and mass numbers of the material, N is Avogadro's number and $r_e = e^2/m_e c^2$ is the classical radius of the electron. C represents the total "area" covered by the electrons contained in one gram, each considered as a sphere of radius r_e.

The parameters that enter in the computation of the collision probability are (§ 4.1): the *mass*, m, of the particle, its *spin* (measured in units of $\hbar$), and its *magnetic moment* (measured in units of $e\hbar/2mc$). We shall assume, however, that the magnetic moment has in all cases the "normal" value; namely, 0 for particles of spin 0, and 1 for charged particles of spin ½ or 1.† In what follows, β represents the velocity of the incident particle in terms of the velocity of light.

(a) *Negative Electrons* (*Negatons*). The collision probability for negatons with negatons has been calculated by Møller (MC32) on the basis of the Dirac theory. When the energy E of the primary particle is large compared with $m_e c^2$ (and therefore $\beta = 1$), Φ_{col} is given by the following expression:

$$\Phi_{col}(E,E')\,dE' = 2Cm_e c^2\,dE' \left[\frac{E}{E'(E-E')} - \frac{1}{E}\right]^2, \tag{2}$$

or

$$\Phi_{col}(E,E')\,dE' = 2C \frac{m_e c^2 E^2\,dE'}{(E-E')^2(E')^2}\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^2. \tag{2a}$$

Since one cannot distinguish between the primary and the secondary particle after the collision, Eq. (2) must be interpreted as giving the probability of a collision that leaves one negaton in the energy state E' and the other in the energy state $E - E'$. Thus one takes into account all possible cases by letting E' vary from 0 to $E/2$ (not E). Equation (2) is symmetrical in E' and $E - E'$.

(b) *Positive Electrons* (*Positons*). Bhabha (BHI36) has calculated the collision probability for positons with negatons. For $E \gg m_e c^2$:

$$\Phi'_{col}(E,E')\,dE' = 2C \frac{m_e c^2\,dE'}{(E')^2}\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^2. \tag{3}$$

* Notice that the probability Φ of a certain interaction, measured in cm^2 g^{-1}, is related to the atomic cross-section, σ, for the same interaction measured in cm^2 by the equation: $\Phi = N\sigma/A$.

† In this sense protons and neutrons have anomalous magnetic moments (see § 4.4).

This expression represents the probability of a collision that gives rise to a secondary negaton of energy in dE' at E'. The probability for a collision out of which the colliding positon comes with an energy in dE' at E' is:

$$\Phi''_{col}(E,E')\,dE' = 2C\,\frac{m_ec^2\,dE'}{(E-E')^2}\left[1-\frac{E'}{E}+\left(\frac{E'}{E}\right)^2\right]^2, \tag{4}$$

as one can easily see by substituting $E - E'$ for E' in Eq. (3). Thus the total probability for a positon-negaton collision after which *either* the negaton *or* the positon has an energy in dE' at E' is:

$$\Phi_{col}(E,E')\,dE' = [\Phi'(E,E') + \Phi''(E,E')]dE',$$

or

$$\Phi_{col}(E,E')\,dE' = 2C\,\frac{m_ec^2E^2\,dE'}{(E-E')^2E'^2}\left[1-\frac{E'}{E}+\left(\frac{E'}{E}\right)^2\right]^2\left[1-2\frac{E'}{E}+2\left(\frac{E'}{E}\right)^2\right]. \tag{5}$$

This expression is analogous to that of Eq. (2a), which gives the collision probability between two negatons. Here again, as in Eq. (2a) one takes into account all possible cases by letting E' vary from 0 to $E/2$ (FFL49). The difference between Eq. (2a) and Eq. (5), expressed by the additional factor:

$$\left[1-2\frac{E'}{E}+2\left(\frac{E'}{E}\right)^2\right],$$

in Eq. (5) arises from the fact that exchange phenomena have different effects in a negaton-negaton and in a positon-negaton collision.

(c) *Particles of Mass m and Spin* 0. Bhabha (BHJ38) has calculated the collision probability for particles of mass m and spin 0:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\,\frac{dE'}{(E')^2}\left(1-\beta^2\frac{E'}{E'_m}\right). \tag{6}$$

(d) *Particles of Mass m and Spin* $\frac{1}{2}$. The collision probability for particles of mass m and spin $\frac{1}{2}$ has been calculated by Bhabha (BHJ38) and by Massey and Corben (MHJ39). It is:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\,\frac{dE'}{(E')^2}\left[1-\beta^2\frac{E'}{E'_m}+\frac{1}{2}\left(\frac{E'}{E+mc^2}\right)^2\right]. \tag{7}$$

(e) *Particles of Mass m and Spin* 1. The collision probability for particles of mass m and spin 1 has been calculated by Massey and Corben (MHJ39) and by Oppenheimer, Snyder, and Serber (OJR40). It is:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\,\frac{dE'}{(E')^2}\left[\left(1-\beta^2\frac{E'}{E'_m}\right)\left(1+\frac{1}{3}\frac{E'}{E_c}\right)+\frac{1}{3}\left(\frac{E'}{E+mc^2}\right)^2\left(1+\frac{1}{2}\frac{E'}{E_c}\right)\right]. \tag{8}$$

where:
$$E_s = \frac{m^2c^2}{m_e}. \tag{9}$$

Note that when E' is very small compared with the maximum transferable energy and with E_s, Eqs. (2), (5), (6), (7), and (8) reduce to the following expression, known as the *Rutherford formula*:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}. \tag{10}$$

Thus, at the limit for small values of E', the collision probabilities of different kinds of particles become identical and depend only on the energy, E', of the secondary electron and on the velocity, β, of the primary particle.

As long as E' is small compared with both E and E_s, Eqs. (7) and (8) reduce to (6) which means that, in this case, the collision probability of a heavy particle is independent of the spin. The difference between the collision probabilities of particles of different spin becomes appreciable when E' is comparable with E_s or with E, a condition that can occur only when E itself is larger than E_s (see Eq. 2.2.4). For these large values of E', the collision probability is an increasing function of the spin. However, the difference between spin $\frac{1}{2}$ and spin 1 is much larger than the difference between spin 0 and spin $\frac{1}{2}$. Let us consider, in particular, the case $E' \ll E'_m$. The collision probabilities for spin 0 and spin $\frac{1}{2}$ follow from the Rutherford formula (10), while the collision probability for spin 1 becomes:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}\left(1 + \frac{1}{3}\frac{E'}{E_s}\right). \tag{11}$$

This expression contains an additional term that decreases with increasing energy as $1/E'$, whereas the Rutherford term decreases as $(1/E')^2$. When $E' > 3E_s$ the additional term, which represents the interaction due to the spin, becomes larger than the Rutherford term, which represents the Coulomb interaction.

Note that the influence of the spin on the collision probability manifests itself only for very close collisions. The theoretical predictions depend essentially on the hypothesis that the electromagnetic field of the particle can be described in the ordinary way even at distances smaller than 10^{-13} cm from the "center" of the particle. So far, this hypothesis lacks any experimental support, so that the validity of the formulae expressing the probabilities of large energy transfers cannot yet be considered as soundly established.

2.4. *Classical derivation of Rutherford's formula.* We have pointed out in the preceding section that at the limit for small values of E', the collision probabilities of all particles with unit charge approach the expression given by Rutherford's formula (2.3.10). In order to illustrate the physical significance of this formula, we shall present, in this section, a derivation based on classical mechanics.

We shall begin by considering a problem of a more general nature than the one discussed so far; namely the problem of a particle of mass m, charge ze and velocity βc interacting electrically with a particle of mass m' and charge $z'e$ at rest. We shall restrict our considerations to the case of small momentum transfers between the two particles, so that, in particular, we may neglect the motion of the target particle during the interaction.

Let b represent the *impact parameter*, i.e., the distance of the line of motion of the incident particle from the target particle before the encounter. Under the assumptions made, b also represents the minimum distance of approach of the two particles. The force between the two particles reaches its maximum value at the moment of closest approach. If we ignore the relativistic deformation of the field (Appendix 2d) for the present, the maximum value of this force is:

$$f = \frac{zz'e^2}{b^2}. \tag{1}$$

Let us first carry out the computation of the momentum transfer in a semi-quantitative way, which, however, brings out the significant physical features of the phenomenon. The "collision time" during which the value of the force is of the same order of magnitude as the maximum value given by Eq. (1) (say greater than $f/2$) is:

$$\tau = \frac{2b}{\beta c}. \tag{2}$$

Hence, the target particles acquires a momentum of the order of:

$$p' = f\tau = \frac{2zz'e^2}{b\beta c}. \tag{3}$$

For reasons of symmetry, this momentum is perpendicular to the trajectory of the incident particle.

In the case of relativistic velocities, the maximum value, f, of the force exerted by the particle on the electron is increased by a factor $1/\sqrt{1-\beta^2}$ over the value given by Eq. (1):

$$f = \frac{zz'e^2}{b^2}\frac{1}{\sqrt{1-\beta^2}}. \tag{4}$$

On the other hand, the "collision time" τ, is decreased by a factor $\sqrt{1-\beta^2}$:

$$\tau = \frac{2b}{\beta c}\sqrt{1-\beta^2}. \tag{5}$$

Thus the product $f\tau$, which gives the momentum acquired by the electron, remains unchanged and Eq. (3) still holds.

A rigorous (classical) proof of Eq. (3) can be given as follows. Consider a cylinder with axis along the trajectory of the moving particle and radius equal to the impact parameter b (Fig. 1). Assume, as before, that the trajectory of the moving particle is not appreciably affected by the collision and that the target particle does not move appreciably during the collision. Let the positive X-axis be in the direction of motion of the particle and let $\mathcal{E}_y$ be the component of the electric field of the moving particle normal to the surface of the cylinder. Since the particle is moving in the direction

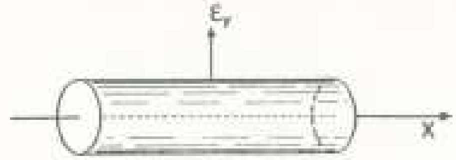

Fig. 2.4.1. Illustrating the derivation of the momentum transfer from a moving particle to a particle at rest.

of increasing X with velocity βc, $\mathcal{E}_y$ depends on the coordinate X and on the time t through a function of the form:

$$\mathcal{E}_y = \mathcal{E}_y(X - \beta ct). \tag{6}$$

From the symmetry properties of the field of a moving charge, one concludes that the resultant momentum, p', acquired by the target particle during the collision is perpendicular to the surface of the cylinder and has the magnitude:

$$p' = z'e \int_{-\infty}^{+\infty} \mathcal{E}_y(X - \beta ct)\, dt. \tag{7}$$

One may transform the integral with respect to t for a fixed X into an integral with respect to X for a fixed t, as follows:

$$\int_{-\infty}^{+\infty} \mathcal{E}_y(X - \beta ct)\, dt = \frac{1}{\beta c} \int_{-\infty}^{+\infty} \mathcal{E}_y(X - \beta ct)\, dX. \tag{8}$$

Application of Gauss's theorem to the integral on the right hand side yields:

$$2\pi b \int_{-\infty}^{+\infty} \mathcal{E}_y(X - \beta ct)\, dX = 4\pi ze. \tag{9}$$

By combining Eqs. (7), (8) and (9) one obtains:

$$p' = \frac{2zz'e^2}{b\beta c}.$$

This expression for p' is identical with that given by Eq. (3).

If one makes the assumption that the kinetic energy, E', acquired by the target particle is small compared with its rest energy, one can compute E' from the nonrelativistic relation between energy and momentum, obtaining:

$$E' = \frac{(p')^2}{2m'} = \frac{2z^2z'^2e^4}{m'c^2b^2\beta^2}. \tag{10}$$

A particle on traversing matter collides with electrons (for which $z' = 1$, $m' = m_e$) and with nuclei (for which $z' = Z$, $m' \approx AM$; M represents here the proton mass). Since there are Z electrons in each atom and since $A \approx 2Z$, Eq. (10) shows that the mean energy transfer to electrons is to the mean energy transfer to nuclei in the ratio of $(Z/m_e)/(Z/2M) = 2M/m_e \approx 4000$. Thus, as far as the energy loss is concerned, collisions with nuclei have a negligible effect compared with collisions with electrons and in this section we need to consider only the latter.

If the target particle is an electron, Eq. (10) may be rewritten as follows:

$$E' = 2m_ec^2 \frac{z^2 r_e^2}{\beta^2 b^2}, \tag{11}$$

where $r_e = e^2/m_ec^2$ is the *classical radius of the electron*.

The probability of an energy transfer in dE' at E' in a given thickness of material is equal to the probability of a collision with an impact parameter in db at b, where E' and b are related by Eq. (11). The probability of a collision with impact parameter in db at b in a thickness of dx g cm^{-2} is given by the expression:

$$F(b)\, db\, dx = 2\pi b\, db\, N \frac{Z}{A}\, dx, \tag{12}$$

where N is Avogadro's number, Z is the charge number of the material through which the particle travels, and A is the corresponding mass number. Differentiation of Eq. (11) yields in absolute value the relation:

$$2b\, db = 2m_ec^2 \frac{z^2}{\beta^2} r_e^2 \frac{dE'}{(E')^2}. \tag{13}$$

By combining Eqs. (12) and (13) one finds the following expression for the probability of an energy loss in dE' at E' on traversal of a thickness dx:

$$\Phi_{col}(E')\, dE'\, dx = \frac{2Cm_ec^2z^2}{\beta^2} \frac{dE'}{(E')^2}\, dx, \tag{14}$$

where C is given by Eq. (2.3.1). Equation (14), with $z = 1$, is identical with Eq. (2.3.10).

The derivation of Rutherford's formula presented above brings out the physical basis for the dependence of $\Phi_{col}(E')$ on the various factors in Eq. (14). The factor C expresses the proportionality of the collision probability to the electron density. The factor $1/\beta^2$ expresses the dependence of the energy transfer on the collision time, and the factor z^2 expresses the dependence of the energy transfer on the strength of the electric interaction between the particle and the electron. The factor $1/(E')^2$ expresses the fact that collisions with large impact parameters are more likely than collisions with small impact parameters. The collision probability does not contain any factor depending on the relativistic deformation of the electric field of the moving particle because this deformation produces two mutually compensating effects, namely, an increase in the field strength and a decrease in the collision time.

The restrictive assumptions underlying the computation of the collision probability and the use of classical mechanics instead of quantum mechanics place limits to the validity of the results obtained. One of the assumptions made is that the electrons are free. Actually they are bound to atoms and can be considered as free only if the collision time is short compared with their period of revolution. If, instead, the collision time is long compared with the period of revolution, the electrons react *adiabatically* to the slowly varying field of the passing particle and do not absorb energy from this field. Let b_1 represent the impact parameter corresponding to a collision time equal to the period of revolution, $T = 1/\nu$, of the atomic electrons. From Eq. (5) one obtains for b_1 the expression:

$$b_1 = \frac{\beta c}{2\pi\nu\sqrt{1-\beta^2}} \tag{15}$$

The arguments developed above show that the expression for the energy transfer E' Eq. (11), loses its validity when the impact parameter is of the order of or greater than b_1.

Likewise it is clear that Eq. (11) must break down for very small impact parameters. According to this equation, E' tends to infinity as b tends to zero. Actually, of course, E' cannot become larger than the maximum transferable energy E'_m defined by Eq. (2.2.4). Moreover, Eq. (11) loses its validity when E' becomes of the order of m_ec^2. This is so because the derivation of Eq. (11) is based upon non-relativistic mechanics; the relativistic correction, that becomes important as E' approaches m_ec^2, causes E' to increase with decreasing b less rapidly than Eq. (11) would indicate. The condition $E' < m_ec^2$ is more restrictive than the condition $E' < E_{max}$, at least if the incident particle has relativistic velocity. It places the following approximate lower limit for the impact parameter:

$$b_2 = \frac{z}{\beta} r_e. \tag{16}$$

The condition $E' < m_ec^2$ is also more restrictive than the conditions implied in neglecting the deflection of the incident particle and the motion of the electron during the collision. The reader can easily prove that if the incident particle has relativistic velocity, these conditions set a lower limit for the impact parameter of the order of:

$$b_3 = 2zr_e\sqrt{1-\beta^2}.$$

If $1 - \beta \ll 1$, then $b_3 \ll b_2$. Therefore one may consider the inequalities:

$$b_1 \gg b \gg b_2 \tag{17}$$

as the classical conditions for the validity of Eq. (11). It is interesting to note that the lower limit of the impact parameter, b_2, is of the order of the classical electron radius.

Quantum-mechanical arguments introduce new limitations to the validity of Eq. (11). The uncertainty principle sets limits to the accuracy that can be achieved in "aiming" a projectile at a given target. Classical mechanics provides an adequate description of a collision process only if the impact parameter is large compared with the "aiming error." Let b_q represent the minimum value of the aiming error. In order to evaluate b_q, consider the motion of the incident particle and of the electron in the center-of-mass system. In this frame of reference the two particles have equal and opposite momenta. If p_0 is the absolute value of the momenta and b the impact parameter, the angular momentum in the center of mass system is p_0b. The angular momentum is conjugate to the angular coordinate. If no restriction is imposed upon the initial position of the incident particle, the angular coordinate has an uncertainty of the order of unity and the angular momentum has an uncertainty of the order of $\hbar$. The corresponding uncertainty, b_q, in the impact parameter is given by the equation:

$$b_q p_0 = \hbar. \tag{18}$$

If the incident particle is an electron, Eq. (18) together with Eq. (A.2.7) in the Appendix yields:

$$b_q = \frac{\sqrt{2}\hbar}{\sqrt{m_e c p}} = \frac{\sqrt{2}\hbar}{m_e c}(1 - \beta^2)^{1/4}. \tag{19}$$

If the mass, m, of the incident particle is very large compared with the mass of the electron, the center of mass of the two particles coincides practically with the incident particle. In this case $p_0 = (m_e/m)p$ and Eq. (18) gives the following expression for b_q:

$$b_q = \left(\frac{\hbar}{m_e c}\right)\left(\frac{mc}{p}\right) = \frac{\hbar\sqrt{1 - \beta^2}}{m_e c \beta}. \tag{20}$$

The length $\hbar/m_e c$ is $\hbar c/e^2 = 137$ times the classical radius of the electron. Therefore the limitation to the impact parameter imposed by the uncertainty principle is more strict than the limitation imposed by classical considerations, Eq. (16), unless the momentum p of the incident particle is very large compared with mc.

2.5. ***Energy loss by collision (ionization loss).*** A charged particle moving through matter loses energy as a consequence of collisions with atomic electrons. In the computation of the collision loss, it is convenient to consider distant collisions and close collisions separately. We shall classify as a distant collision any collision that results in the ejection of an electron of energy smaller than a predetermined value, η. We shall classify as a close collision any collision that results in the ejection of an electron of energy larger than η. If the limiting energy η is sufficiently small (and the corresponding impact parameter sufficiently large) we can treat all distant collisions by considering the primary particle as a point charge. If the limiting energy η is sufficiently large (and the corresponding impact parameter sufficiently small) we can treat all close collisions by considering the atomic electrons as free particles. For practically all cases of importance in the field of high-energy phenomena, a limiting energy between 10^4 and 10^5 ev simultaneously satisfies both conditions specified above. In what follows we shall assume that the limiting energy lies within this range.

Let $k_{\mathrm{coll}(<\eta)}(E)$ be the energy loss per g cm^{-2} resulting from distant collisions. In the computation of $k_{\mathrm{coll}(<\eta)}(E)$ it is essential to take into account the binding of the electrons to the atoms; i.e., one should consider the system formed by an atom and by the incident particle and then compute the probabilities for the various possible transitions leading to excitation or ionization of the atom. Bethe (BHA30; BHA32) developed a theory along these lines. With the help of Born's approximation, and for the case of particles with unit charge, he obtained the following result:

$$k_{\mathrm{coll}(<\eta)}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{2m_ec^2\beta^2\eta}{(1-\beta^2)I^2(Z)} - \beta^2\right], \tag{1}$$

where $I(Z)$ is the average ionization potential of an atom of atomic number Z.

The quantity $I(Z)$ can be evaluated theoretically, or it can be deduced from experimental data. Bloch (BF33) suggested the formula:

$$I(Z) = I_HZ, \tag{2}$$

where $I_H = 13.5$ is the energy corresponding to the Rydberg frequency. More accurate calculations were carried out by Wick (WGC41; WGC43) and by Halpern and Hall (HO48). Table 1 summarizes the various determinations of I. The discrepancies between these determinations reflect

Table 1. Values of the average ionization potential of various substances

Substance	Z	Author	Method	I (ev)
Hydrogen	1	Bethe (BHA30)	Theoretical	14.9
Helium	2	Williams (WEJ37)	Theoretical	35
		Halpern and Hall (HO48)	Theoretical	40
Carbon	6	Wick (WGC43)	Theoretical	60
		Halpern and Hall (HO48)	Theoretical	60
Aluminum	13	Wilson (WRR41)	Experimental	150
Iron	26	Wick (GC43)	Theoretical	243
		Halpern and Hall (HO48)	Theoretical	430
Gold	79	Livingston and Bethe (LMS37)	Experimental	520
Lead	82	Wick (WGC41)	Experimental	1000
		Halpern and Hall (HO48)	Theoretical	1200
Air		Livingston and Bethe (LMS37)	Experimental	80.5
		Halpern and Hall (HO48)	Theoretical	96
Water		Wick (WGC41)	Theoretical (hydrogen) Experimental (oxygen)	63
		Halpern and Hall (HO48)	Theoretical	80

the present uncertainty as to the actual values of the average ionization potential. This uncertainty, however, does not represent a serious source of error in the computations of $k_{col(<\eta)}(E)$ because I enters only in the logarithm.

Equation (1) is valid for particles of any kind, with positive or negative charge equal to e and *with velocity large compared with the velocity of atomic electrons.*

Consider next the energy loss per g cm^{-2} resulting from close collisions, i.e., from collisions in which the energy transfer is greater than η. This quantity shall be called $k_{col(>\eta)}(E)$. In the computation of $k_{col(>\eta)}(E)$, one may consider the electrons as free. One thus obtains the following expression:

$$k_{col(>\eta)}(E) = \int_{\eta}^{E'_m} E'\Phi_{col}(E,E')\,dE', \tag{3}$$

where E'_m is the maximum transferable energy [see Eq. (2.2.4)].

(a) *Heavy Particles.* For singly charged particles heavier than electrons and with energy small compared with m^2c^2/m_e one can use Eq. (2.3.6) which gives (if $\eta \ll E'_m$):

$$k_{col(>\eta)}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{E'_m}{\eta} - \beta^2\right]. \tag{4}$$

The total energy loss by collision per g cm^{-2} (or *ionization loss*):

$$k_{col}(E) = -\frac{dE}{dx} \tag{5}$$

is the sum of $k_{col(<\eta)}$ and $k_{col(>\eta)}$ and has the expression:

$$k_{col}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{2m_ec^2\beta^2E'_m}{(1-\beta^2)I^2(Z)} - 2\beta^2\right]. \tag{6}$$

This expression is independent of the arbitrary value chosen for the limiting energy η, as it should be. Substituting E'_m from Eq. (2.2.7) transforms Eq. (6) into the following:

$$k_{col}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{4m_e^2c^4\beta^4}{(1-\beta^2)^2I^2(Z)} - 2\beta^2\right]. \tag{7}$$

Within the limit of validity of Eq. (2.2.7), k_{col} is only a function of β, i.e., of the velocity of the incident particle. Since $p/mc = \beta/\sqrt{1-\beta^2}$, one may also say that k_{col} does not depend separately on the momentum and on the mass of the incident particle, but only on the ratio of these two quantities. Likewise, one may say that k_{col} does not depend separately on the energy and the mass of the incident particle, but only on their ratio. The same is true of the quantity $k_{col(<\eta)}$ and, in this case, without the restrictive condition (2.2.6) that insures the validity of Eq.

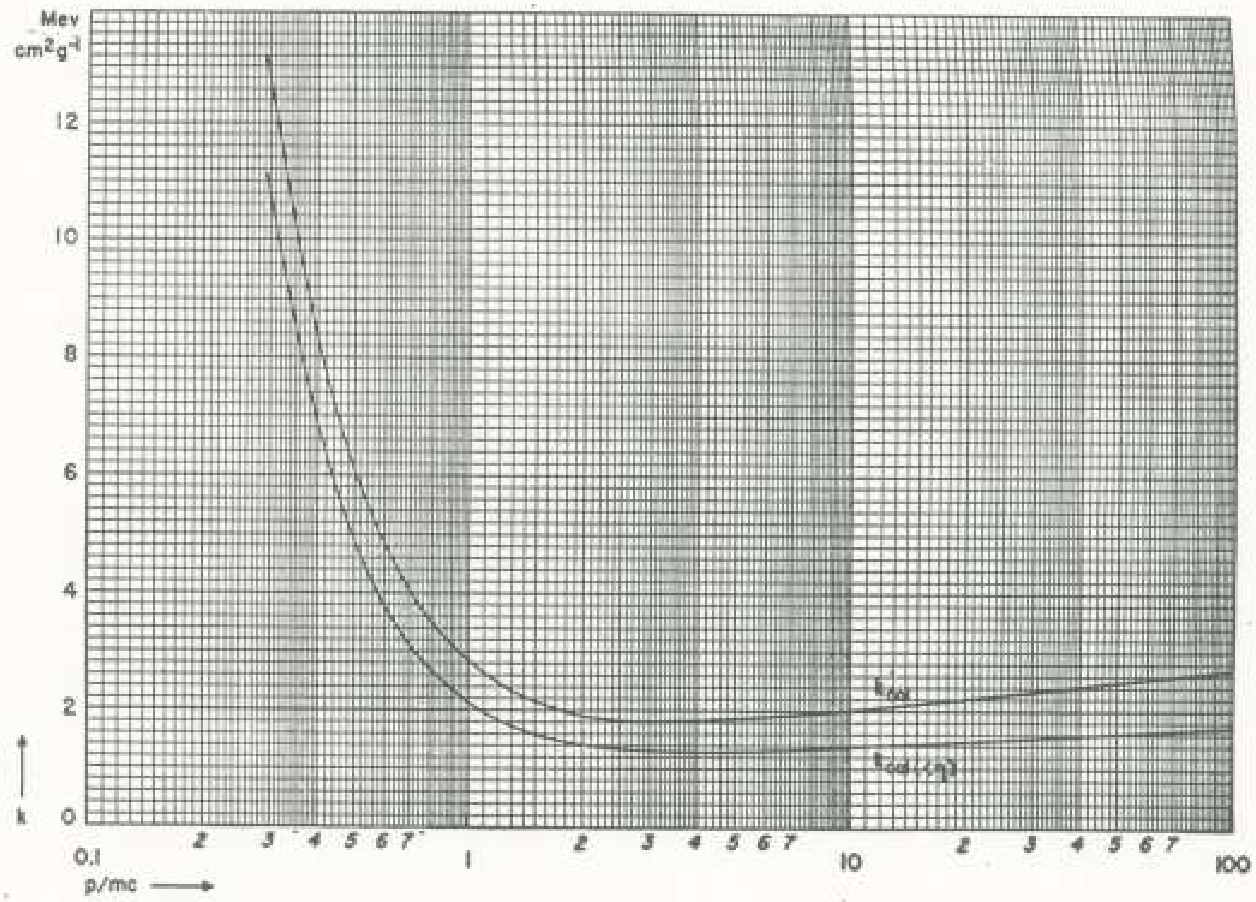

Fig. 2.5.1. The total collision loss, k_{col}, and the energy loss in distant collisions, $k_{col(<\eta)}$, for particles heavier than electrons in air, as functions of p/mc ($\eta = 10^4$ ev). $k_{col(<\eta)}$ was computed from Eq. (2.5.1); k_{col} was obtained from a paper by Smith (SJH47).

(2.2.7). The functional dependence of $k_{col(<a)}$ and k_{col} on p/mc for air is illustrated in Fig. 1.

In order to appreciate the physical significance of Eqs. (6) or (7) we shall derive an approximate expression for the collision loss of heavy particles by means of semi-classical considerations.

From Eq. (2.4.11) one finds, for a singly charged particle, that the energy loss per g cm^{-2} due to collisions with impact parameter in db at b has the expression:

$$2\pi b\, db\, N \frac{Z}{A} E'(b) = \frac{4Cm_ec^2}{\beta^2}\frac{db}{b} \tag{8}$$

It has been shown in § 2.4 that, if the energy of the incident particle is not very large compared with its rest energy, Eq. (2.4.11) is valid when $b_1 > b > b_2$, where b_1 and b_2 are given by Eqs. (2.4.15) and (2.4.20) respectively. For impact parameters larger than b_1 or smaller than b_2, Eq. (2.4.11) overestimates the energy transfer. Thus one may evaluate the total energy loss by integrating the expression (8) between b_2 and b_1:

$$k_{col}(E) = \frac{4Cm_ec^2}{\beta^2}\int_{b_2}^{b_1}\frac{db}{b} = \frac{4Cm_ec^2}{\beta^2}\ln\frac{b_1}{b_2}, \tag{9}$$

or, from Eqs. (2.4.15) and (2.4.20):

$$k_{col}(E) = \frac{2Cm_ec^2}{\beta^2}\ln\frac{\pi^2 m_e^2 c^4 \beta^4}{(1-\beta^2)^2 h^2\nu^2}. \tag{10}$$

If one substitutes $I(Z)$ for $h\nu$ in Eq. (10), one obtains an expression for k_{col} that does not differ significantly from Eq. (7).

Despite this agreement, one should not take the classical picture too literally. For example, the classical treatment does not give the correct number of energy transfers, nor their correct distribution in space. One can easily recognize this fact by considering that already for impact parameters considerably smaller than b_1 the "classical" energy transfer, E', as given by Eq. (2.4.11), is smaller than the excitation energy of the atoms. Only when one computes the total energy loss by integrating the classical expression over all impact parameters (and neglects the impossibility of energy transfers smaller than the excitation energy) does one obtain a correct result.

Figure 1 shows that, for subrelativistic energies, the energy loss, k_{col}, decreases rapidly with increasing energy because of the term β^2 in the denominator. This term arises from the similar term in Eq. (2.4.14) and corresponds to the fact that, for a given impact parameter, the interaction between the passing particle and the atom becomes less effective as the time spent by the particle near the atom becomes shorter. When β approaches its limiting value of 1, the factor $1/\beta^2$ becomes practically constant; k_{col} goes through a flat minimum at a momentum equal to a small multiple of mc and then begins to increase with increasing momentum because of the factor $1/(1-\beta^2)^2$ in the logarithm. The reason for this increase is twofold: (1) as the velocity increases, the relativistic deformation of the Coulomb field of the incident particle causes the effects of this particle to be felt at larger distances from its geometric path and therefore increases the upper limit of the impact parameter [see Eq. (2.4.15)]; (2) as the momentum increases, the quantum-theoretical uncertainty, which sets the lower limit of the impact parameter, decreases [see Eq. (2.4.20)].

The dependence on momentum of $k_{col(<\epsilon)}$ is very similar to that of k_{col}. In the relativistic region, however, $k_{col(<\epsilon)}$ increases with p somewhat more slowly than k_{col}. The physical reason for this is that in the case of $k_{col(<\epsilon)}$, the lower limit of the impact parameter is determined by the limiting energy and does not vary with p. Thus the increase with momentum is caused exclusively by the effect of the relativistic deformation of the Coulomb field on the upper limit of the impact parameter.

(b) *Electrons.* The total energy loss of *negatons* and *positons* can be calculated easily from Eqs. (2.3.2), (2.3.3), (1), and (3). If one remembers that $\beta \approx 1$, one obtains:

$$k_{col} = 2Cm_ec^2\left[\ln\left(\frac{\pi^2(m_ec^2)^2}{(1-\beta^2)^{3/2}I^2(Z)}\right) - a\right], \tag{11}$$

where $a = 2.9$ for negatons; $a = 3.6$ for positons.

Here again we may justify the theoretical expression for the energy loss by semi-classical considerations. Indeed Eq. (9) together with Eqs. (2.4.15) and (2.4.19) gives (since $\beta = 1$):

$$k_{col} = 2Cm_ec^2 \ln \frac{\pi^2(m_ec^2)^2}{2(1-\beta^2)^{3/2}h^2\nu^2}. \tag{12}$$

Equation (12) is very similar to Eq. (11). Note in both equations the term: $-\ln(1-\beta^2)^{3/2}$ that gives the dependence of the collision loss of electrons on velocity and compare it with the term: $-\ln(1-\beta^2)^2$ that gives the dependence on velocity of the collision loss of heavy particles. The derivation of Eqs. (10) and (12) shows that the difference arises from the different relation between the momenta of the incident particle in the center-of-mass system and in the laboratory system respectively.

The expression (2.4.11) for the energy transfer shows that the collision loss of a particle with multiple charge, ze, is z^2 times the collision loss of a particle with unit charge and the same velocity.

The *momentum* loss is easily obtained from the *energy* loss. Indeed, since $dp/dE = 1/\beta c$, the following simple relation holds:

$$-\frac{d(pc)}{dx} = -\frac{1}{\beta}\frac{dE}{dx} = \frac{k_{col}}{\beta}. \tag{13}$$

The momentum loss is a function of the velocity alone whenever this is true of the energy loss.

Some measurements of the collision loss of particles heavier than electrons will be discussed in § 6.4.

2.6. ***The density effect.*** So far, in investigating the interactions of charged particles with atoms, we have considered the latter as isolated. This is permissible to a large extent when the particle travels in a gas. When the particle travels in a condensed material we can still consider the atoms as isolated in the case of close collisions, but we cannot do so

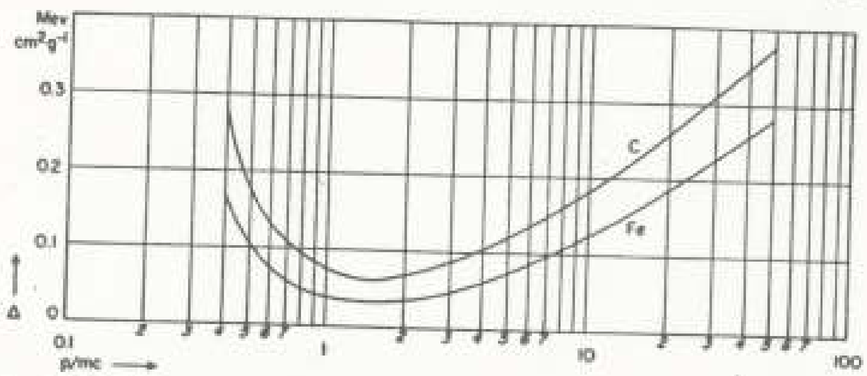

Fig. 2.6.1. The decrease in collision loss, Δ, due to density effect as a function of p/mc, for carbon and iron. From Wick (WGC43).

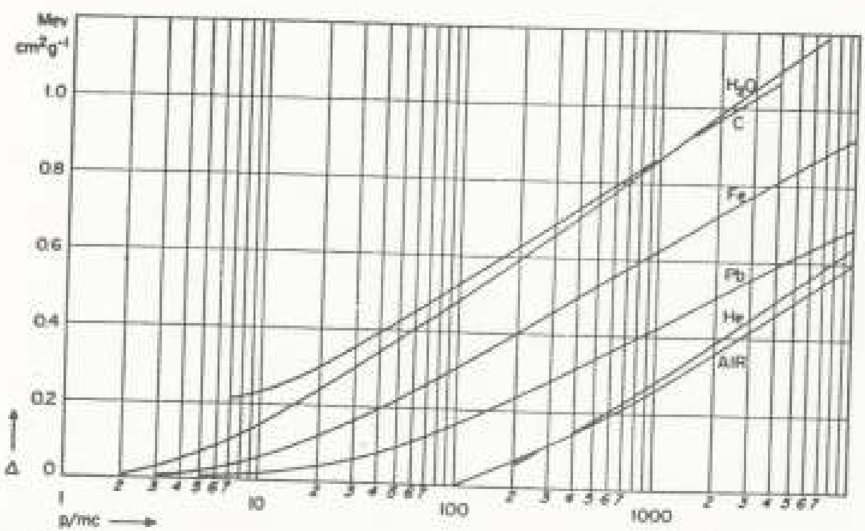

Fig. 2.6.2. The decrease in collision loss, Δ, due to density effect as a function of p/mc, for water, carbon, iron, lead, helium and air. From Halpern and Hall (HO48).

when the impact parameter is larger than the atomic distances. For such distant collisions one has to take into account the screening of the electric field of the passing particle by the atoms of the medium. The screening reduces the interaction and decreases, therefore, the energy loss. Since distant collisions become more and more important as the velocity increases, the correction to be applied to the expression for the energy loss is an increasing function of the velocity. The influence of the density on the collision loss was first suggested by Swann (SWF38) and quantitatively investigated by Fermi (FE39). According to Fermi, the quantity Δ to be subtracted from the energy loss, as calculated for isolated atoms, is given by the following formulae, in the case of singly charged particles:

$$\text{for} \qquad \beta < \epsilon^{-1/2}, \quad \Delta(\beta) = \frac{2Cm_ec^2}{\beta^2} \ln \epsilon;$$

$$\text{for} \qquad \beta > \epsilon^{-1/2}, \quad \Delta(\beta) = \frac{2Cm_ec^2}{\beta^2} \left[\ln \frac{\epsilon - 1}{1 - \beta^2} + \frac{1 - \epsilon\beta^2}{\epsilon - 1} \right]; \tag{1}$$

where ϵ is the dielectric constant of the medium relative to vacuum.

Halpern and Hall (HO40; HO48) and Wick (WGC41; WGC43) made a more refined analysis of the density effect by considering in detail the behavior of atomic electrons belonging to the different shells. Their computations confirmed the finding that the collision loss depends on the density of the absorbing material, but showed that the simplification made by Fermi in the development of the theory lead, in general, to an overestimate of the reduction in the collision loss.

Figure 1 represents the results of Wick's calculations for carbon and iron. Figure 2 represents the results of the calculations of Halpern and Hall for carbon, water, iron, lead, air, and helium. One sees that the agreement between the two sets of data, where they can be compared, leaves much to be desired.

The energy loss of charged particles in materials of finite density has been studied further by A. Bohr (BLA48), by Messel and Ritson (MH50.2), and by Schönberg (SbM51). These investigators called attention to the fact that part of the energy dissipated by high-energy particles in their interactions with atomic electrons goes into electromagnetic radiation (*Cerenkov* radiation) rather than into excitation or ionization of atoms. The intensity of the Cerenkov radiation (which, of course, must not be confused with the radiation that accompanies the deflection of the incident particle in the electric fields of nuclei) increases with increasing velocity. Indeed, it appears that the relativistic increase of the energy loss by distant collisions is mainly due to the increase of the Cerenkov radiation.

2.7. ***Statistical fluctuations in the energy loss by collision.*** The energy loss of a charged particle in matter is a statistical phenomenon because the collisions that are responsible for this loss are independent events. Thus particles of a given kind and of a given energy do not all lose exactly the same amount of energy in traversing a given thickness of material. The quantity $k_{col}(E)$ defined as "collision loss" in § 2.5 represents only an average value. The statistical fluctuations in the energy loss by collision are comparatively small because the average transfer of energy in each individual collision process is small and the number of collisions necessary to cause any appreciable energy change is correspondingly large.

For electrons, in general, collision processes are not the main cause of energy losses and especially not the main cause of fluctuations in the